On Rationalizability in Two-Person Alternating-Offer Bargaining

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October, 1996

Abstract

This paper reconsiders Rubinstein's alternating-offer bargaining game with complete information. We define rationalizability for multi-stage games with observed actions. We show that rationalizability does not exclude perpetual disagreement or delay. Then, we define trembling-hand rationalizability and we show that it implies a unique solution. Moreover, this unique solution is the subgame perfect equilibrium. We also reconsider an extension of Rubinstein's game wherein there is a smallest money unit.

JEL Classification: C72, C78.
Keywords: bargaining, alternating-offer, rationalizability, money unit.

*Parts of this paper were presented at seminars and conferences at the University of Bonn, University of Louvain, Hebrew University of Jerusalem, Unviversity of Valencia. The author thanks the participants for their comments and suggestions. A special thanks goes to Pierpaolo Battigalli, Eric van Damme, Claude d'Aspremont, Pierre Dehez, Martin Dufwenberg, Françoise Forges and Hans Peters for helpful comments and discussions. The usual disclaimer applies.

†This Research has been supported by a grant "Actions de Recherche Concertées" 93/98-162 of the Ministery of Scientific Research of the Belgian French Speaking Community.

1 Introduction

> ... we do gain some insight into the behavior of Homo sapiens by studying Homo rationalis.
>
> - Robert J. Aumann, *Frontiers of Economics*

"Is uniqueness in Rubinstein's bargaining model robust to weaker solution concepts?" This paper reconsiders Rubinstein's [13] alternating-offer bargaining game with complete information. Rubinstein has shown that there is a unique partition of the cake, which can be supported as a subgame perfect equilibrium (SPE). In this SPE, agreement is reached without delay and the less impatient player obtains a larger share of the cake. A notion like SPE assumes common expectations of players' behaviour. That is, each player holds a correct conjecture about her opponent's strategy (or actions) choice. But once we admit the possibility that a player may have several strategies (or actions) that she could reasonably use, conjectures and strategies (or actions) actually played may be mismatched. This is what distinguishes rationalizability (Bernheim [4], and Pearce [11]) from equilibrium concepts.

To offer an answer to our opening question, we will solve Rubinstein's bargaining game using rationalizability for multi-stage games with observed actions. Fudenberg and Tirole [7] argued that rationalizability implies a unique solution to Rubinstein's bargaining game. In this chapter we show that in fact perpetual disagreement is even possible. Then we define trembling-hand rationalizability for our multi-stage bargaining game with observed actions. Trembling-hand rationalizability is a weak refinement of rationalizability that still preserves the basic characteristics of the non-equilibrium approach. We show that there is a unique solution to Rubinstein's bargaining game that is trembling-hand rationalizable. Moreover this unique solution is the SPE of the game.

Using the rationalizability concepts, we also reconsider an extension of Rubinstein's game, developed by van Damme *et al.* [15] and Muthoo [9], wherein there is a smallest money unit (i.e. the number of feasible agreements is finite). Van Damme *et al.* [15] and Muthoo [9] have shown that any Pareto-efficient division (belonging to the finite set of feasible divisions) can be supported as a SPE of Rubinstein's game, provided that the period between successive offers is sufficiently small. Once we adopt the non-equilibrium approach, we obtain a similar result. In this paper, we also give a necessary and sufficient condition such that there is a unique solution to this bargaining game that is trembling-hand rationalizable.

Recent work on extensive-form rationalizability (EFR) includes Battigalli [3], Börgers

[5], and Schuhmacher [14][1]. Other papers have used weaker solution concepts for solving bargaining games (see Cho [6], and Watson [17]). These papers deal with bargaining games with incomplete information. Watson [17] studied Rubinstein's bargaining game with two-sided incomplete information about the players' discount factors. But he also stated that his solution concept, iterated conditional dominance, doesn't lead to a unique outcome and doesn't exclude perpetual disagreement in the complete information case as well. For solving a one-sided offer bargaining model under one-sided incomplete information (where a seller makes an offer in each period and a buyer has private information about his reservation value), Cho [6] used subgame rationalizability (Bernheim [4]) but with the restriction that the buyer must use weakly stationary strategies and that this restriction is common knowledge.

The paper is organized as follows. In Section 2, the bargaining model is presented. Section 3 is devoted to rationalizability. In Section 4 we give the result on rationalizability. Section 5 investigates a refinement of rationalizability: trembling-hand rationalizability. In Section 6 we reconsider an extension of Rubinstein's model wherein a smallest money unit is introduced. Section 7 concludes.

2 The Bargaining Model

Two players i $(i = 1, 2)$ are bargaining over the division of a cake of size one. These two players must agree on an allocation from the set X. X is called the set of possible agreements.

$$X \equiv \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0,\ x_1 + x_2 \leq 1\right\}$$

We denote by x_i player i's share, for $i = 1, 2$. Consider an alternating-call bargaining procedure. Player 1 calls (offers / accepts) in even-numbered periods and player 2 calls in odd-numbered periods. Let $n \in \mathbb{N}$ be the period at which an offer is made. The game starts at $n = 0$ and ends when one of the players accepts the opponent's previous offer. Note that an agreement may be reached as early as in period $n = 1$. In each period n, the player on the move chooses an action $a(n) \in A \equiv X \cup \{\text{accept}\}$; except at the very beginning of the game where player 1 cannot accept. Let $h^0 = \emptyset$ be the history at the start of play. Define an history of the game at the end of period $k-1$ by $h^k = (a(0), ..., a(k-1)) \in H^k \equiv \prod_{n=0}^{k-1} A$. Given h^k and $l < k$ $(l, k \in \mathbb{N})$, we call h^l a sub-history of h^k if h^l is the

[1] Battigalli [3] studied Pearce's definition of EFR which was defined as an elimination procedure. He showed that EFR can be characterized in terms of restriction on players' updating system. Börgers [5] studied the trembles approach to EFR. Schuhmacher [14] developed a rationalizability concept for extensive-form games which takes into account that rationality cannot be common belief at every information set in the game.

first l elements of h^k, and we write $h^l < h^k$. Histories after which player 1 has the move are contained in $H_1 \equiv \bigcup_{n=0}^{\infty} H^{2n}$. Histories after which player 2 has the move are contained in $H_2 \equiv \bigcup_{n=0}^{\infty} H^{2n+1}$. Let $H \equiv H_1 \cup H_2$. A pure strategy of player i is a function $s_i : H_i \to A$ which maps each possible history after which player i has the move into an action. Let S_i be the set of strategies for player i, $i = 1, 2$. $-i$ denotes player i's opponent. $S \equiv S_1 \times S_2$ is the set of strategy profiles. Payoffs in the bargaining game are given as functions of the players' strategy profile according to the vN-M utility functions $U_i : S \to \mathbb{R}$, $i = 1, 2$; where $U_i(s) \equiv u_i(\theta(s))$. For any outcome $\theta(s) \equiv ((x_1(s), x_2(s)), n(s)) \in X \times \mathbb{N}$ that specifies an agreement on allocation $(x_1(s), x_2(s))$ at period $n(s)$, without too much loss of generality[2], let $u_i(\theta(s)) \equiv \delta_i^{n(s)-1} x_i(s)$, where $\delta_i \in (0,1)$ is player i's discount factor, for $i = 1, 2$. If under s the players fail to reach an agreement, let $u_i(\theta(s)) = 0$, for $i = 1, 2$. Denote by $\mathrm{G}(\delta_1, \delta_2)$ the alternating-offer bargaining game of complete information. We denote by $\triangle(A)$ the set of probability distributions on A. A behaviour strategy for player i, denoted $c_i = (c_i(h))_{h \in H_i}$, specifies a probability distribution over actions after each $h \in H_i$, $c_i(h) \in \triangle(A)$; and the probability distributions after different histories are independent. We interpret behaviour strategies of player i as a way of describing player $-i$'s beliefs about player i's actions for each history after which she has to move; $c_i(h) \in \triangle(A)$ represents a belief of player $-i$ concerning player i's action after history h. Let C_i be the set of behaviour strategies for player i, $i = 1, 2$; and let $C \equiv C_1 \times C_2$. Given any set $\widehat{S}_i \subseteq S_i$, $i = 1, 2$, let $C_i\left(\widehat{S}_i\right)$ denote the set of behaviour strategies of player i when her set of pure strategies is $\widehat{S}_i$. Note that a pure strategy is a special kind of behaviour strategy in which the distribution after each histories is degenerate. (s_i, c_{-i}) generates a probability distribution over outcomes in the obvious way, and hence gives rise to an expected payoff for each player. Player i's expected payoff given (s_i, c_{-i}) is denoted by $U_i(s_i, c_{-i})$. Given any set $B \subseteq A$, let $\triangle^0(B)$ be the set of all *non-degenerate* probability distributions on B. That is, if B is not a singleton, $\triangle^0(B)$ is the set of all probability distributions that do not attach probability one to a single action in the set B. A *non-degenerate* behaviour strategy for player i (or non-degenerate conjecture of player $-i$), denoted $\widehat{c}_i = (\widehat{c}_i(h))_{h \in H_i}$, specifies a probability distribution over actions after each history; $\widehat{c}_i(h) \in \triangle^0(A)$ represents a non-degenerate belief of player $-i$ concerning player

[2]Indeed, it would be enough to assume that players' preferences over outcomes satisfy Osborne and Rubinstein's [10] assumptions A1 to A6. These assumptions impose the following conditions on players' preferences over outcomes: (A1) disagreement is the worst outcome, (A2) cake is desirable, (A3) time is valuable, (A4) player i's preference ordering is continuous, (A5) stationarity of player i's preference ordering, and (A6) the larger the share of the cake the larger the compensation player i requires to be indifferent to incur a delay of one period. For ease of exposition, we assume that $u_i(\theta(s)) \equiv \delta_i^{n(s)-1} x_i(s)$. The preferences that this function represents satisfy these assumptions.

i's action after history h. Let $\widehat{C}_i$ be the set of *non-degenerate* behaviour strategies for player i, $i = 1, 2$; and let $\widehat{C} \equiv \widehat{C}_1 \times \widehat{C}_2$. Given any set $\widehat{S}_i \subseteq S_i$, $i = 1, 2$, let $\widehat{C}_i\left(\widehat{S}_i\right)$ denote the set of *non-degenerate* behaviour strategies of player i when her set of pure strategies is $\widehat{S}_i$. Given a history $h \in H$, let $\mathrm{G}(h)$ be the continuation game, i.e. the game that begins after h. Given (s_i, c_{-i}) and a history h, the continuation strategy profile is the one induced by (s_i, c_{-i}) in $\mathrm{G}(h)$. Given $h \in H^k$, we denote by $U_i(s_i, c_{-i} \mid h)$ $[U_i(s_i, \widehat{c}_{-i} \mid h)]$ the expected payoff of player i in the game conditional on h describing the play through period k (or stage k) and (s_i, c_{-i}) $[(s_i, \widehat{c}_{-i})]$ describing the play thereafter.

3 Rationalizability for Multi-Stage Games

For solving the alternating-offer bargaining game $\mathrm{G}(\delta_1, \delta_2)$, instead of using Pearce's definition of EFR, we define rationalizability (RM) for our multi-stage game with observed actions. RM is based on the following assumptions: (**A1**) both players are rational, (**A2**) **A1** is common knowledge, and (**A3**) the structure of the game (strategy sets, payoffs functions) is common knowledge. Formally, RM is defined by the following iterative process.

Definition 1 *Consider the game $G(\delta_1, \delta_2)$. Let $R^0 \equiv S$. Then $R^k \equiv R_1^k \times R_2^k$ $(k \geq 1)$ is inductively defined as follows: for $i = 1, 2$, $s_i \in R_i^k$ if: (i) $s_i \in R_i^{k-1}$; (ii) $\forall h \in H_i$ $\exists$ $c_{-i} \in C_{-i}\left(R_{-i}^{k-1}\right)$ such that $\forall$ $s_i' \in R_i^{k-1}$, $U_i(s_i, c_{-i} \mid h) \geq U_i(s_i', c_{-i} \mid h)$. The set of rationalizable strategy profiles is $R^\infty \equiv \bigcap_{k=0}^{\infty} R^k$.*

Assumption **A1** means that both bargainers are playing strategies which are rational. In Definition 1, the set R_i^1 is the set of player i's rational strategies. A strategy $s_i \in S_i$ is rational if, after each individual history $h \in H_i$, there exists a conjecture, $c_{-i} \in C_{-i}$, such that it is a best response to this conjecture in $\mathrm{G}(h)$. That is, a rational player maximizes her expected payoff after each sub-history played, given her conjecture (i.e. given her collection of beliefs about her opponent's action for each history after which he has to act).

Assumption **A2** means that it is common knowledge that both bargainers are rational. We call a fact mutual knowledge of order 1 between two players if they both know it, mutual knowledge of order 2 if they both know that both know it, and so on for any order. Let mutual knowledge up to order n be denoted by $\mathrm{mk}(n)$. Thus $\mathrm{mk}(1)$ is mutual knowledge of order 1 and $\mathrm{mk}(\infty)$ is common knowledge. Mutual knowledge of order 1 of rationality means that player i knows that player $-i$ is rational. Mutual knowledge up to order 2 ($\mathrm{mk}(2)$) of rationality means that rationality is $\mathrm{mk}(1)$ and that player i knows that player $-i$ knows that player i is rational. Mutual knowledge up to order 3 ($\mathrm{mk}(3)$) of rationality means that rationality is $\mathrm{mk}(2)$ and that player i knows that player $-i$ knows that player

i knows that player $-i$ is rational; and so on. In Definition 1, the set R^2 relies on the assumption of mutual knowledge of order 1 (mk(1)) of rationality. mk(1) of rationality means that player i should not have arbitrary conjectures or collections of beliefs about player $-i$'s actions for each history after which he has to act. She should hold conjectures, c_{-i}, which belong to $C_{-i}\left(R_{-i}^1\right)$. That is, she should expect her opponent to use only strategies that are rational; she should expect her opponent to use only *rational actions*. Let $s_i\left(h\right)$ be the action specified by the strategy s_i after the history h. Then, an action $a_i \in A$ of player i is rational (after history $h \in H_i$) if and only if there exists some rational strategy, $s_i \in R_i^1$, of player i such that $s_i\left(h\right) = a_i$. In Definition 1, the set R^3 relies on the assumption of mutual knowledge up to order 2 (mk(2)) of rationality. mk(2) of rationality means that player i should hold conjectures, c_{-i}, which belong to $C_{-i}\left(R_{-i}^2\right)$. That is, she should expect her opponent to use only strategies that are best responses to some conjectures (or collections of beliefs) that he might have. These opponent's collections of beliefs shouldn't be arbitrary, it should give weight only on player i's actions that are rational; and so on for higher-order of mutual knowledge of rationality.

Remark that in Definition 1, $\left\{R^k; k \geq 0\right\}$ is a weakly decreasing sequence, i.e. $\emptyset \neq R^{k+1} \subseteq R^k \; \forall k \in \mathbb{N} \cup \{\infty\}$. The limit set is given by $R^\infty \equiv \lim_{k\to\infty} R^k = \bigcap_{k=0}^{\infty} R^k$. R^k denotes the set of pure strategy profiles which survive k round of iteration. Each higher step of the iteration requires a higher-order of mutual knowledge of assumption **A1**. At round k of the elimination procedure, a strategy is said to be rational if it hasn't yet been deleted. Condition (ii) in Definition 1 restricts conjectures or collections of beliefs to the set of behaviour strategies associated with the set of pure strategies that have not been eliminated at a previous stage. Denote by $R^\infty\left(\delta_1, \delta_2\right)$ the set of rationalizable strategy profiles for the bargaining game $\mathrm{G}(\delta_1, \delta_2)$.

4 Main Result

It has been argued in the literature (see Fudenberg and Tirole [7, pp.129-130] and Kreps [8, p.560]) that rationalizability or iterated conditional dominance solves Rubinstein's bargaining game by supporting a unique strategy profile which is the SPE one. Nevertheless, the next Proposition 1 shows that rationalizability does not exclude inefficient outcomes (perpetual disagreement or delay)[3].

[3]Note that Watson [17] mentioned that iterated conditional dominance does not exclude perpetual disagreement or delay and that this result was known to Martin Osborne.

Proposition 1 *Consider the bargaining game* $G(\delta_1, \delta_2)$. *Then, strategy profiles that are rationalizable are not unique: the set* $R^{\infty}(\delta_1, \delta_2)$ *is not a singleton. Moreover, there are strategy profiles* $s \in R^{\infty}(\delta_1, \delta_2)$ *leading to agreements reached with delay or to perpetual disagreement.*

Proof. From Definition 1, $R^0 = S$. $s_i \in R_i^1$ are such that player i will accept (after any history h) all $x_i > \delta_i$, for $i = 1, 2$. Player i with discount factor δ_i is indifferent between receiving $\delta_i x_i$ today and x_i tomorrow. Since $x_i = 1$ is the biggest share player i can obtain, she will accept all $x_i > \delta_i$. Compute R^2. $s_i \in R_i^2$ are such that player i will never offer $x_{-i} > \delta_{-i}$ and will reject all $x_i < \delta_i(1 - \delta_{-i})$, for $i = 1, 2$. Indeed, player i will never accept an offer which gives her less than $\delta_i(1 - \delta_{-i})$, because she could obtain close to $1 - \delta_{-i}$ next period by offering $x_{-i} = \delta_{-i} + \varepsilon$ with ε small. All player i's conjectures, $c_{-i} \in C_{-i}(R_{-i}^1)$, are such that, for all $h \in H_{-i}$, $c_{-i}(h)$ gives probability one on player $-i$'s acceptance of $\delta_{-i} + \varepsilon$; and $\forall \xi \in (0, 1)$ $\exists \varepsilon \in (0, 1)$ such that $\delta_i(1 - \delta_{-i} - \varepsilon) > \delta_i(1 - \delta_{-i}) - \xi$. R^2 tells us that the largest share of the cake player i could obtain in a continuation game, $\mathrm{G}(h \in H_i)$, where she calls first is $1 - \delta_{-i}(1 - \delta_i)$. Compute R^3. $s_i \in R_i^3$ are such that player i will never offer (after any history h) $x_{-i} > \delta_{-i}(1 - \delta_i(1 - \delta_{-i}))$, will reject all $x_i < \delta_i(1 - \delta_{-i}(1 - \delta_i(1 - \delta_{-i})))$ and will accept all $x_i > \delta_i(1 - \delta_{-i}(1 - \delta_i))$. Indeed, player i will never accept an offer which gives her less than $\delta_i(1 - \delta_{-i}(1 - \delta_i(1 - \delta_{-i})))$, because she could obtain close to $1 - \delta_{-i}(1 - \delta_i(1 - \delta_{-i}))$ next period by offering $x_{-i} = \delta_{-i}(1 - \delta_i(1 - \delta_{-i})) + \varepsilon$ with ε small. All player i's conjectures, $c_{-i} \in C_{-i}(R_{-i}^2)$, are such that, for all $h \in H_{-i}$, $c_{-i}(h)$ gives probability one on player $-i$'s acceptance of $\delta_{-i}(1 - \delta_i(1 - \delta_{-i})) + \varepsilon$; and $\forall \xi \in (0, 1)$ $\exists \varepsilon \in (0, 1)$ such that $\delta_i(1 - \delta_{-i}(1 - \delta_i(1 - \delta_{-i})) - \varepsilon) > \delta_i(1 - \delta_{-i}(1 - \delta_i(1 - \delta_{-i}))) - \xi$. Then, R^3 tells us that the largest share of the cake player i could obtain in a continuation game, $\mathrm{G}(h \in H_i)$, where she calls first is $1 - \delta_{-i}(1 - \delta_i(1 - \delta_{-i}(1 - \delta_i)))$; and so on... Compute R^k. Let $\left\{x_i^k\right\}_{k=0}^{\infty}$, $i = 1, 2$, be strict monotonic decreasing sequences defined by $x_i^k \equiv \delta_i\left(1 - \delta_{-i}\left(1 - x_i^{k-1}\right)\right)$ and starting with $x_i^0 \equiv 1$. $s_i \in R_i^k$ are such that player i will: (a) never offer (after any history h) $x_{-i} > x_{-i}^{k-1} \equiv \delta_{-i}\left(1 - \delta_i\left(1 - x_{-i}^{k-2}\right)\right)$, (b) accept all $x_i > x_i^{k-1} \equiv \delta_i\left(1 - \delta_{-i}\left(1 - x_i^{k-2}\right)\right)$, (c) reject all $x_i < \delta_i\left(1 - x_{-i}^{k-1}\right)$. These assertions (a), (b), (c) are easily verified. The strict monotonic decreasing sequences $\left\{x_i^k\right\}_{k=0}^{\infty}$, $i = 1, 2$, converge in the limit to $x_i^{\infty} = \delta_i(1 - \delta_{-i})[1 - \delta_i\delta_{-i}]^{-1}$, $i = 1, 2$. Compute R^{∞}. $s_i \in R_i^{\infty}$ are such that player i will offer (after any history) $x_{-i} = \delta_{-i}(1 - \delta_i)[1 - \delta_i\delta_{-i}]^{-1}$, will accept all $x_i > \delta_i(1 - \delta_{-i})[1 - \delta_i\delta_{-i}]^{-1}$, and will reject all $x_i < \delta_i(1 - \delta_{-i})[1 - \delta_i\delta_{-i}]^{-1}$. It follows that one $s \in R^{\infty}$ is the SPE, where 1 offers (at period $n = 0$) $x_2 = \delta_2(1 - \delta_1)[1 - \delta_1\delta_2]^{-1}$, offer which is accepted by 2 at period $n = 1$ (see

Osborne and Rubinstein [10]). But R^{∞} is not a singleton[4]. 2's expected payoff is equal to $\delta_2(1-\delta_1)[1-\delta_1\delta_2]^{-1}$. Therefore, 2 may reject an offer $(1-\delta_2, \delta_2(1-\delta_1))[1-\delta_1\delta_2]^{-1}$, hoping that his counter-offer $(\delta_1(1-\delta_2), 1-\delta_1)[1-\delta_1\delta_2]^{-1}$ will be accepted by 1 next period; 2 may hold a collection of beliefs, $c_1 \in C_1(R_1^{\infty})$, such that after any history h player 1 will accept his counter-offer. A similar reasoning can be made for player 1. Therefore, players may even play a strategy profile $s \in R^{\infty}$ leading to an agreement reached with delay or to perpetual disagreement. ■

5 Trembling-Hand Rationalizability

The starting point of trembling-hand rationalizability (TRM) for our multi-stage game with observed actions is that the rationality concept is strengthened by asking that a player's strategy be optimal not only given her conjecture but also given perturbed conjectures[5]. TRM is defined by modifying the iterative procedure of Definition 1.

Definition 2 *Consider the game $G(\delta_1, \delta_2)$. Let $T^0 \equiv S$. Then $T^k \equiv T_1^k \times T_2^k$ $(k \geq 1)$ is inductively defined as follows: for $i = 1, 2$, $s_i \in T_i^k$ if: (i) $s_i \in T_i^{k-1}$; (ii) $\forall\ h \in H_i$ $\exists\ \widehat{c}_{-i} \in \widehat{C}_{-i}\left(T_{-i}^{k-1}\right)$ such that $\forall\ s_i' \in T_i^{k-1}$, $U_i(s_i, \widehat{c}_{-i} \mid h) \geq U_i(s_i', \widehat{c}_{-i} \mid h)$. The set of trembling-hand rationalizable strategy profiles is $T^{\infty} \equiv \bigcap_{k=0}^{\infty} T^k$.*

In Definition 2, the set T_i^1 is the set of player i's trembling-hand rational strategies. A strategy, $s_i \in S_i$, is *trembling-hand rational* if, after each individual history $h \in H_i$, there exists some non-degenerate conjecture $\widehat{c}_{-i} \in \widehat{C}_{-i}$ against which s_i is a best response in G(h). At step k of the iteration, a strategy $s_i \in T_i^{k-1}$ belongs to T_i^k if, after each $h \in H_i$, there exists some non-degenerate conjecture $\widehat{c}_{-i} \in \widehat{C}_{-i}\left(T_{-i}^{k-1}\right)$ against which s_i is a best response in G(h). $\left\{T^k; k \geq 0\right\}$ is a weakly decreasing sequence, i.e. $\emptyset \neq T^{k+1} \subseteq T^k$ $\forall k \in \mathbb{N} \cup \{\infty\}$. The limit set is given by $T^{\infty} \equiv \lim_{k \to \infty} T^k = \bigcap_{k=0}^{\infty} T^k$. Let $T^k(\delta_1, \delta_2)$ and $T^{\infty}(\delta_1, \delta_2)$ be, respectively, the set of k-step trembling-hand rationalizable strategy profiles and the set of trembling-hand rationalizable strategy profiles for G(δ_1, δ_2).

Proposition 2 *Consider the bargaining game $G(\delta_1, \delta_2)$. Then, strategy profile that is trembling-hand rationalizable is unique and is the SPE: $s \in T^{\infty}(\delta_1, \delta_2)$ leads to an offer $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ by player 1 at $n = 0$, which is accepted by player 2 at $n = 1$.*

[4] Fudenberg and Tirole [7, pp.129-130] have argued that rationalizability or iterated conditional dominance solves Rubinstein's game by supporting a unique strategy profile which is the SPE one. Nevertheless, they have forgotten that player 1 (2) is indifferent between agreeing on $(\delta_1(1-\delta_2), 1-\delta_1)[1-\delta_1\delta_2]^{-1}$ today (tomorrow) and on $(1-\delta_2, \delta_2(1-\delta_1))[1-\delta_1\delta_2]^{-1}$ tomorrow (today). These indifferences imply that conjectures and strategies (or actions) actually played may be mismatched.

[5] This restriction on the best-response correspondence may be interpreted as if the players have some doubt about the strategies played by their opponent.

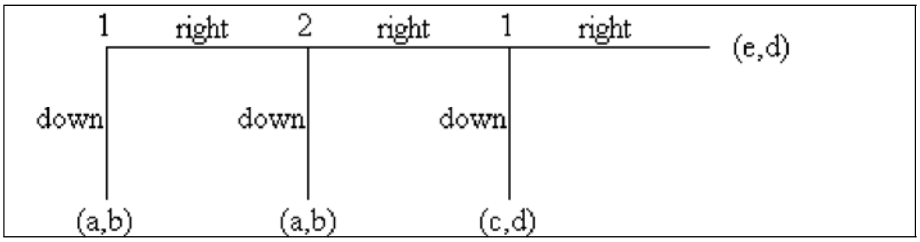

Figure 1: A two-player game

Proof. From Definition 2, $T^0 = S$. $s_i \in T_i^1$ are such that i will accept (after any history h) all $x_i \geq \delta_i$, for $i = 1, 2$. Player i will accept a share of δ_i. Given that she holds a non-degenerate conjecture, she cannot believe with probability one that her opponent will accept $x_{-i} = 0$. Therefore, if she refuses a share of δ_i then her highest expected payoff will be less than δ_i. Take the non-degenerate conjecture where she believes that her opponent will (with probability $1 - p$) accept $x_{-i} = 0$ and will (with probability p) propose $x_i = 1$ when on the move. As $p \in (0, 1)$ approaches zero, her expected payoff approaches δ_i. Therefore, for all $\widehat{c}_{-i} \in \widehat{C}_{-i}$, it is a best response for player i to accept (after any history h) all $x_i \geq \delta_i$. Moreover, it is immediate that player i's strategies which reject an offer $\delta_i x_i$ today and propose x_i tomorrow are excluded from the set T_i^1. From the proof of Proposition 1 and the argumentation here above, we have that $s_i \in T_i^k$ are such that player i will never offer (after any history h) $x_{-i} > x_{-i}^{k-1} \equiv \delta_{-i}\left(1 - \delta_i\left(1 - x_{-i}^{k-2}\right)\right)$, will accept all $x_i \geq x_i^{k-1} \equiv \delta_i\left(1 - \delta_{-i}\left(1 - x_i^{k-2}\right)\right)$, and will reject all $x_i < \delta_i\left(1 - x_{-i}^{k-1}\right)$. In the limit, $s_i \in T_i^\infty$ is such that player i will offer (after any history h) $x_{-i} = x_{-i}^\infty = \delta_{-i}(1 - \delta_i)[1 - \delta_i\delta_{-i}]^{-1}$, will accept all $x_i \geq x_i^\infty = \delta_i(1 - \delta_{-i})[1 - \delta_i\delta_{-i}]^{-1}$, and will reject all $x_i < \delta_i(1 - x_{-i}^\infty) = \delta_i(1 - \delta_{-i})[1 - \delta_i\delta_{-i}]^{-1}$. This unique strategy profile $s \in T^\infty$ is such that 1 offers (at $n = 0$) $x_2 = \delta_2(1 - \delta_1)[1 - \delta_1\delta_2]^{-1}$, offer which is accepted by 2 at $n = 1$. Therefore, $s \in T^\infty$ is the SPE (see Osborne and Rubinstein [10]). The outcome (efficient) of the game is the agreement $(1 - \delta_2, \delta_2(1 - \delta_1))[1 - \delta_1\delta_2]^{-1}$ which is also players' payoffs of the game. ■

For some games the commonality of the knowledge that players are *trembling-hand rational* runs into problems. The following example illustrates this inconsistency.

Consider a simple two-player game whose extensive-form is depicted in Figure 1. Player 1 has two decision nodes. At each decision node she has two actions, "down" and "right". Player 2 has one decision node where he has two actions, "down" and "right". Every terminal node yields a payoff for both players. The first number is the payoff for player 1 and the second number the payoff for player 2. Payoffs are such that a>b>c>d>e. Player 1 has four pure strategies; $S_1 = \{$(down,down),(down,right),(right,down),(right,right)$\}$. Player 2 has two pure strategies; $S_2 = \{$down,right$\}$. It is quite obvious that $T_1^1 = T_1^\infty = \{$(down,down)$\}$ and $T_2^1 = T_2^\infty = \{$down$\}$. Mutual knowledge of order 1 of *trembling-hand rationality* means that player 1 knows that player 2 will play "down". Therefore, player 1 is indifferent in opening the game by playing "down" or "right". Nevertheless the opening action "right" is not trembling-hand rationalizable. The problem[6] is: why should player 1 plays a trembling-hand rationalizable strategy if player 1 knows that player 2 will play a trembling-hand rationalizable strategy? Let $\overline{T}^0 \equiv S$. Then $\overline{T}^k \equiv \overline{T}_1^k \times \overline{T}_2^k$ $(k \geq 1)$ is inductively defined as follows: for $i = 1, 2$, $s_i \in \overline{T}_i^k$ if: (i) $s_i \in S_i$; (ii) $\forall\ h \in H_i \quad \exists\ \widehat{c}_{-i} \in \widehat{C}_{-i}\left(T_{-i}^{k-1}\right)$ such that $\forall\ s_i' \in S_i$, $U_i\left(s_i, \widehat{c}_{-i} \mid h\right) \geq U_i\left(s_i', \widehat{c}_{-i} \mid h\right)$. Thus each set $\overline{T}_i^k$ consists of unconstrained best responses, while each set T_i^k consists of constrained best responses. Remark that in our simple two-person game: $\overline{T}_1^2 = \{$(down,down),(right,down)$\} \supset T_1^\infty = \overline{T}_1^1$ and $\overline{T}_2^2 = T_2^\infty = \{$down$\}$. Nevertheless, for the alternating-offer bargaining game $\mathrm{G}(\delta_1, \delta_2)$ we have that, for all $k \in \mathbb{N}$, $\overline{T}_i^k = T_i^k$ $(i = 1, 2)$.

6 Finitely Many Divisions of the Cake

Using the rationalizability concepts defined in Sections 3 and 5, we reconsider an extension of the alternating-offer bargaining game, developed by van Damme *et al.* [15] and Muthoo [9], wherein they introduced a smallest money unit. That is, the bargaining problem becomes the division of a cake (or a fixed amount of money) which can be shared only in finitely many different alternatives (i.e. there is a smallest money unit). Then, van Damme *et al.* [15] and Muthoo [9] have shown that any partition of the cake can be supported by a SPE if the period between successive offers is sufficiently small. Adopting the non-equilibrium approach, it is quite obvious that we will not rule out the multiplicity of solutions. Nevertheless, our approach leads to bounds on the players' payoffs or shares.

6.1 Bargaining with a Smallest Money Unit

Both players i $(i = 1, 2)$ are still bargaining over the division of a cake of size one or an amount of money normalized to one. Let $g > 0$ be the smallest money unit. Then, the set of possible agreements becomes

$$X \equiv \{(n_1 g, n_2 g) \mid n_i \in \mathbb{N}, (n_1 + n_2)\, g \leq 1\} \tag{1}$$

[6]The inconsistency problem has been studied by Reny [12]. To resolve such an inconsistency of common knowledge of *trembling-hand* rationality, Asheim and Dufwenberg [1] have changed the object for the common knowledge: instead of common knowledge of rational choice, they assume common knowlledge of rational *reasoning*.

Let X^e denotes the set of efficient agreements, i.e. $(n_1 + n_2)\, g = 1$. The negotiation proceeds according to the rules defined in Section 2. The sets of actions, histories and strategies are adapted, in a obvious way, to be fitted for this alternative bargaining game. Denote by $\mathrm{G}(\delta_1, \delta_2, g)$ the alternating-offer bargaining game with a smallest money unit. We can also express the bargainer i's discount factor in terms of discount rate $r_i \in (0,1)$, by the formula $\delta_i = \exp(-r_i \Delta)$, where Δ is the length of the bargaining period. Greater patience implies a lower discount rate and a higher discount factor: $r_1 \geq r_2 \Leftrightarrow \delta_1 \leq \delta_2$. Let $R^\infty(\delta_1, \delta_2, g)$ and $T^\infty(\delta_1, \delta_2, g)$ be, respectively, the set of rationalizable strategy profiles and the set of trembling-hand rationalizable strategy profiles in $\mathrm{G}(\delta_1, \delta_2, g)$. Denote by $Y \equiv \{n_i g \mid n_i \in \mathbb{N}, 0 \leq n_i g \leq 1\}$ player i's possible share of the cake in $\mathrm{G}(\delta_1, \delta_2, g)$. In order to derive the results of Section 6.2, we define two functions $f : [0,1] \rightarrow Y$ and $F : [0,1] \rightarrow Y$, where $\forall\, y \in [0,1]$, $f(y) \equiv y - \mathrm{mod}[y, g]$ and $F(y) \equiv y - \mathrm{mod}[y, g] + g$. Note that $\mathrm{mod}[y, g]$ means y modulo g, i.e. the remainder from dividing y by g.

6.2 Results on Rationalizability

Next we state some properties of the sets $R^\infty(\delta_1, \delta_2, g)$ and $T^\infty(\delta_1, \delta_2, g)$ for our bargaining game $\mathrm{G}(\delta_1, \delta_2, g)$ of complete information.

Proposition 3 *Consider a bargaining game $G(\delta_1, \delta_2, g)$. If δ_1, δ_2, and g are such that*

$$\begin{cases} (1-g) \leq \delta_1 \\ (1-g) \leq \delta_2 \end{cases} \tag{2}$$

then $\forall\, x \in X \quad \exists\, s \in R^\infty(\delta_1, \delta_2, g)$ such that $\theta(s) = (x, n)$.

Proof. From Definition 1, $R^0 = S$. $s_i \in R_i^1$ are such that player i will accept (after any history h) all $x_i \geq F(\delta_i) = \delta_i - \mathrm{mod}[\delta_i, g] + g$, for $i = 1, 2$. Player i with discount factor δ_i is indifferent between receiving $\delta_i x_i$ today and x_i tomorrow. Since $x_i = 1$ is the biggest share player i can obtain, she will accept all $x_i \geq \delta_i - \mathrm{mod}[\delta_i, g] + g$. Compute R^2. $s_i \in R_i^2$ are such that player i will never offer: $x_{-i} > F(\delta_{-i}) = \delta_{-i} - \mathrm{mod}[\delta_{-i}, g] + g$ if $\mathrm{mod}[\delta_{-i}, g] \neq 0$; $x_{-i} > f(\delta_{-i}) = \delta_{-i}$ if $\mathrm{mod}[\delta_{-i}, g] = 0$. $s_i \in R_i^2$ are such that player i will reject all: $x_i \leq f(\delta_i(1 - F(\delta_{-i})))$ if $\mathrm{mod}[\delta_i(1 - F(\delta_{-i})), g] \neq 0$ and $\mathrm{mod}[\delta_{-i}, g] \neq 0$; $x_i \leq f(\delta_i(1 - f(\delta_{-i})))$ if $\mathrm{mod}[\delta_i(1 - F(\delta_{-i})), g] \neq 0$ and $\mathrm{mod}[\delta_{-i}, g] = 0$; $x_i < f(\delta_i(1 - F(\delta_{-i})))$ if $\mathrm{mod}[\delta_i(1 - F(\delta_{-i})), g] = 0$ and $\mathrm{mod}[\delta_{-i}, g] \neq 0$; $x_i < f(\delta_i(1 - f(\delta_{-i})))$ if $\mathrm{mod}[\delta_i(1 - F(\delta_{-i})), g] = 0$ and $\mathrm{mod}[\delta_{-i}, g] = 0$. Consider the case where $\mathrm{mod}[\delta_i(1 - F(\delta_{-i})), g] \neq 0$ and $\mathrm{mod}[\delta_{-i}, g] \neq 0$. Then, player i will never accept an offer which gives her less or equal than $f(\delta_i(1 - F(\delta_{-i})))$, because she could obtain $1 - F(\delta_{-i})$ next period by offering $x_{-i} = F(\delta_{-i})$. All player i's conjectures,

$c_{-i} \in C_{-i}\left(R^1_{-i}\right)$, are such that, for all $h \in H_{-i}$, $c_{-i}(h)$ gives probability one on player $-i$'s acceptance of $F(\delta_{-i})$.

R^2 tells us that the largest share of the cake player i could obtain in a continuation game where she calls first is: $1 - F(\delta_{-i}(1 - F(\delta_i)))$ if $\text{mod}\left[\delta_i(1 - F(\delta_{-i})), g\right] \neq 0$ and $\text{mod}\left[\delta_{-i}, g\right] \neq 0$; $1 - F(\delta_{-i}(1 - f(\delta_i)))$ if $\text{mod}\left[\delta_i(1 - F(\delta_{-i})), g\right] \neq 0$ and $\text{mod}\left[\delta_{-i}, g\right] = 0$; $1 - f(\delta_{-i}(1 - F(\delta_i)))$ if $\text{mod}\left[\delta_i(1 - F(\delta_{-i})), g\right] = 0$ and $\text{mod}\left[\delta_{-i}, g\right] \neq 0$; $1 - f(\delta_{-i}(1 - f(\delta_i)))$ if $\text{mod}\left[\delta_i(1 - F(\delta_{-i})), g\right] = 0$ and $\text{mod}\left[\delta_{-i}, g\right] = 0$. It follows that if $F(\delta_i) = 1$ $(i = 1, 2)$ then all $s_i \in R^k_i$ $(k \in \mathbb{N})$ are such that player i will never offer $x_{-i} > 1$ and will reject all $x_i < 0$, $i = 1, 2$. Then, it is straightforward that, if $F(\delta_i) = 1$ $(i = 1, 2)$ then $\forall\ x \in X \quad \exists\ s \in R^\infty(\delta_1, \delta_2, g)$ such that $\theta(s) = (x, n)$. Remark that $F(\delta_i) = 1$ can be rewritten as $(1 - g) = \delta_i - \text{mod}\left[\delta_i, g\right]$ which is equivalent to $(1 - g) \leq \delta_i$; indeed, $\left[F(\delta_i) = 1\right] \Leftrightarrow \left[(1 - g) \leq \delta_i\right]$, $(i = 1, 2)$. ■

Since $\delta_i = \exp(-r_i\Delta)$ for $i = 1, 2$, this proposition tells us that if the length of a single bargaining round is sufficiently small (i.e. if the negotiator's speed of response becomes faster) then any (feasible) partition of the cake (or amount of money) is rationalizable. Moreover, if $(1 - g) \leq \delta_i$ $(i = 1, 2)$, then for any efficient agreement $x \in X^e$ there exists a rationalizable strategy profile $s \in R^\infty(\delta_1, \delta_2, g)$ that results in the outcome $\theta(s) = (x, 1)$. Proposition 3 is displayed by the following example.

Example 1: Player 1 gets the entire sum of money.

Let $\delta_1 = \delta_2 = .9$, $g = .2$, $x = (1, 0)$, and $n = 1$. Assume that players hold the following rationalizable conjectures: player 1 holds $c_2 = (c_2(h))_{h \in H_2} \in C_2(R^\infty_2)$ where, $\forall\ h \in H_2$, $c_2(h)$ gives probability one on player 2's acceptance of $x_2 \geq 0$; player 2 holds $c_1 = (c_1(h))_{h \in H_1} \in C_1(R^\infty_1)$ where, $\forall\ h \in H_1$, $c_1(h)$ gives probability one on player 1's acceptance of $x_1 = 1$ and rejection of $x_1 < 1$. Given their conjectures, it is a best response for player 1 to offer $x = (1, 0)$ at $n = 0$, and for player 2 to accept this offer at $n = 1$. Then, $\theta = ((1, 0), 1)$ is the outcome (efficient) of the negotiation implemented at $n = 1$. Nevertheless, given these conjectures, it is also a best response for player 2 to reject $x = (1, 0)$ and to propose $x = (1, 0)$ at $n = 1$. Then, the inefficient outcome (reached with delay) $\theta = ((1, 0), 2)$ is implemented.

Next we are looking for a condition that implies that $\mathrm{G}(\delta_1, \delta_2, g)$ is solvable by rationalizability for multi-stage games. From now on, without too much loss of generality, we assume that $\delta_i = \delta$ $(i = 1, 2)$ where $\delta \in (0, 1)$. Then, a necessary condition is the following one.

$$(1 - g) > \delta \tag{3}$$

Van Damme *et al.* [15] have shown that in the equilibrium approach such a condition is not sufficient to guarantee a unique SPE for $\mathrm{G}(\delta, g)$. Indeed, they showed that if

$\text{mod}\left[\frac{1}{1+\delta}, g\right] < g\left(\frac{1-\delta}{1+\delta}\right)$ and $\frac{1}{1+\delta}$ is not an integer multiple of g, then the game $\mathrm{G}(\delta, g)$ has a unique SPE with outcome

$$\theta = \left(\left(\frac{1}{1+\delta} - \text{mod}\left[\frac{1}{1+\delta}, g\right], \frac{\delta}{1+\delta} + \text{mod}\left[\frac{1}{1+\delta}, g\right]\right), 1\right)$$

Next we give three examples where the condition (3) is satisfied, meanwhile the condition $\text{mod}\left[\frac{1}{1+\delta}, g\right] < g\left(\frac{1-\delta}{1+\delta}\right)$ is not satisfied in Example 2.

Example 2: A non-solvable bargaining game.

Let $\delta = .85$ and $g = .1$. Then, $s_i \in R_i^1$ are such that player i will accept all $x_i \geq F(\delta) = .9$, $i = 1, 2$. $s_i \in R_i^2$ are such that player i will never offer $x_{-i} > F(\delta) = .9$ and will reject all $x_i \leq f(\delta(1 - F(.85))) = 0$. $s_i \in R_i^3$ are such that player i will never offer $x_{-i} > F(\delta(1 - F(\delta(1 - .9)))) = .8$ and will reject all $x_i \leq f(\delta(1 - .8)) = .1$. $s_i \in R_i^4$ are such that player i will never offer $x_{-i} > F(\delta(1 - F(\delta(1 - .8)))) = .7$ and will reject all $x_i \leq f(\delta(1 - .7)) = .2$. $s_i \in R_i^5$ are such that player i will never offer $x_{-i} > F(\delta(1 - F(\delta(1 - .7)))) = .6$ and will reject all $x_i \leq f(\delta(1 - .6)) = .3$. $s_i \in R_i^k$ $(k \geq 6)$ are such that player i will never offer $x_{-i} > F(\delta(1 - F(\delta(1 - .6)))) = .6$ and will reject all $x_i \leq f(\delta(1 - .6)) = .3$. It follows that player i's rationalizable strategies are such that player i will accept all offer $x_i \geq .6$, will never offer $x_{-i} > .6$, and will reject all $x_i \leq .3$. Therefore, rationalizable strategies are not unique. Note that since $\text{mod}\left[\frac{1}{1+\delta}, g\right] > g\left(\frac{1-\delta}{1+\delta}\right)$ we cannot state that the SPE is unique.

Example 3: A solvable bargaining game.

Let $\delta = .66$ and $g = .05$. Then, $s_i \in R_i^1$ are such that player i will accept all $x_i \geq F(\delta) = .7$, $i = 1, 2$. $s_i \in R_i^2$ are such that player i will never offer $x_{-i} > F(\delta) = .7$ and will reject all $x_i \leq f(\delta(1 - F(.66))) = .15$. $s_i \in R_i^3$ are such that player i will never offer $x_{-i} > F(\delta(1 - F(\delta(1 - .7)))) = .55$ and will reject all $x_i \leq f(\delta(1 - .55)) = .25$. $s_i \in R_i^4$ are such that player i will never offer $x_{-i} > F(\delta(1 - F(\delta(1 - .55)))) = .5$ and will reject all $x_i \leq f(\delta(1 - .55)) = .3$. $s_i \in R_i^5$ are such that player i will never offer $x_{-i} > F(\delta(1 - F(\delta(1 - .5)))) = .45$ and will reject all $x_i \leq f(\delta(1 - .45)) = .35$. $s_i \in R_i^k$ $(k \geq 6)$ are such that player i will never offer $x_{-i} > F(\delta(1 - F(\delta(1 - .45)))) = .4$ and will reject all $x_i \leq f(\delta(1 - .4)) = .35$. It follows that player i's rationalizable strategies are such that player i will accept all offer $x_i \geq .4$, will never offer $x_{-i} > .4$, and will reject all $x_i \leq .35$. Therefore, as well as the SPE, there is a unique rationalizable strategy profile that results in the outcome $\theta = ((.6, .4), 1)$. It is also the unique SPE outcome of the game.

Example 4: Two SPE payoffs.

Let $\delta = \frac{2}{3}$ and $g = .05$. Then, $s_i \in R_i^1$ are such that player i will accept all $x_i \geq F(\delta) = .7$, $i = 1, 2$. $s_i \in R_i^2$ are such that player i will never offer $x_{-i} > F(\delta) = .7$ and will reject all $x_i < f(\delta(1 - f(.7))) = .2$. $s_i \in R_i^3$ are such that player i will never offer $x_{-i} > F(\delta(1 - f(\delta(1 - .7)))) = .55$ and will reject all $x_i < f(\delta(1 - .55)) = .3$. $s_i \in R_i^4$ are such that player i will never offer $x_{-i} > F(\delta(1 - f(\delta(1 - .55)))) = .5$ and will reject all $x_i \leq f(\delta(1 - .5)) = .3$. $s_i \in R_i^5$ are such that player i will never offer

$x_{-i} > F(\delta(1 - F(\delta(1 - .5)))) = .45$ and will reject all $x_i \leq f(\delta(1 - .45)) = .35$. $s_i \in R_i^k$ $(k \geq 6)$ are such that player i will never offer $x_{-i} > F(\delta(1 - F(\delta(1 - .45)))) = .45$ and will reject all $x_i \leq f(\delta(1 - .45)) = .35$. It follows that player i's rationalizable strategies are such that player i will accept all offer $x_i \geq .45$, will never offer $x_{-i} > .45$, and will reject all $x_i \leq .35$. Therefore, rationalizable strategies are not unique, but the condition $\bmod\left[\frac{1}{1+\delta}, g\right] < g\left(\frac{1-\delta}{1+\delta}\right)$ is satisfied.

Regarding the last example, van Damme *et al.* [15] have shown that whenever Rubinstein's SPE partition $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ is a multiple of g and $\delta + g \leq 1$, then there exists two SPE payoff vectors, $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ and $\left(\frac{1}{1+\delta} - g, \frac{\delta}{1+\delta} + g\right)$. That is, the SPE of G($\delta$) being a SPE of G($\delta, g$) is not a sufficient condition for uniqueness. Let $\overline{z}$ be the solution of

$$\overline{z} \equiv \begin{cases} F(\delta(1 - F(\delta(1 - \overline{z})))) & \text{if } \begin{cases} \bmod[\delta(1 - F(\delta(1 - \overline{z}))), g] \neq 0 \\ \bmod[\delta(1 - \overline{z}), g] \neq 0 \end{cases} \\ F(\delta(1 - f(\delta(1 - \overline{z})))) & \text{if } \begin{cases} \bmod[\delta(1 - F(\delta(1 - \overline{z}))), g] \neq 0 \\ \bmod[\delta(1 - \overline{z}), g] = 0 \end{cases} \\ f(\delta(1 - F(\delta(1 - \overline{z})))) & \text{if } \begin{cases} \bmod[\delta(1 - F(\delta(1 - \overline{z}))), g] = 0 \\ \bmod[\delta(1 - \overline{z}), g] \neq 0 \end{cases} \\ f(\delta(1 - f(\delta(1 - \overline{z})))) & \text{if } \begin{cases} \bmod[\delta(1 - F(\delta(1 - \overline{z}))), g] = 0 \\ \bmod[\delta(1 - \overline{z}), g] = 0 \end{cases} \end{cases} \tag{4}$$

and $\underline{z}$ be the solution of

$$\underline{z} = \begin{cases} \delta(1 - \overline{z}) - \bmod[\delta(1 - \overline{z}), g] + g & \text{if } \bmod[\delta(1 - \overline{z}), g] \neq 0 \\ \delta(1 - \overline{z}) & \text{if } \bmod[\delta(1 - \overline{z}), g] = 0 \end{cases} \tag{5}$$

Lemma 1 *Consider a bargaining game G(δ, g). If δ is such that $(1 - g) > \delta$, then $s \in R^\infty(\delta, g)$ and $s \in T^\infty(\delta, g)$ are such that player i $(i = 1, 2)$ will never offer $x_{-i} > \overline{z}$, and will reject $x_i < \underline{z}$.*

Proof. Assume that $\delta_i = \delta \in (0, 1)$, $(i = 1, 2)$, and $(1 - g) > \delta$. The sets R^1 and R^2 have been computed in the proof of Proposition 3. Compute R^3. $s_i \in R_i^3$ are such that player i will never offer (after any history h):
$x_{-i} > F(\delta(1 - F(\delta(1 - F(\delta)))))$ if $\bmod[\delta(1 - F(\delta(1 - F(\delta)))), g] \neq 0$, $\bmod[\delta, g] \neq 0$, $\bmod[\delta(1 - F(\delta)), g] \neq 0$;
$x_{-i} > f(\delta(1 - F(\delta(1 - F(\delta)))))$ if $\bmod[\delta(1 - F(\delta(1 - F(\delta)))), g] = 0$, $\bmod[\delta, g] \neq 0$, $\bmod[\delta(1 - F(\delta)), g] \neq 0$;
$x_{-i} > F(\delta(1 - F(\delta(1 - f(\delta)))))$ if $\bmod[\delta(1 - F(\delta(1 - f(\delta)))), g] \neq 0$, $\bmod[\delta, g]$

$= 0$, $\mathrm{mod}\left[\delta\left(1 - F\left(\delta\right)\right), g\right] \neq 0$;
$x_{-i} > f\left(\delta\left(1 - F\left(\delta\left(1 - f\left(\delta\right)\right)\right)\right)\right)$ if $\mathrm{mod}\left[\delta\left(1 - F\left(\delta\left(1 - f\left(\delta\right)\right)\right)\right), g\right] = 0$, $\mathrm{mod}\left[\delta, g\right]$ $= 0$, $\mathrm{mod}\left[\delta\left(1 - F\left(\delta\right)\right), g\right] \neq 0$;
$x_{-i} > F\left(\delta\left(1 - f\left(\delta\left(1 - F\left(\delta\right)\right)\right)\right)\right)$ if $\mathrm{mod}\left[\delta\left(1 - f\left(\delta\left(1 - F\left(\delta\right)\right)\right)\right), g\right] \neq 0$, $\mathrm{mod}\left[\delta, g\right]$ $\neq 0$, $\mathrm{mod}\left[\delta\left(1 - F\left(\delta\right)\right), g\right] = 0$;
$x_{-i} > f\left(\delta\left(1 - f\left(\delta\left(1 - F\left(\delta\right)\right)\right)\right)\right)$ if $\mathrm{mod}\left[\delta\left(1 - f\left(\delta\left(1 - F\left(\delta\right)\right)\right)\right), g\right] = 0$, $\mathrm{mod}\left[\delta, g\right]$ $\neq 0$, $\mathrm{mod}\left[\delta\left(1 - F\left(\delta\right)\right), g\right] = 0$;
$x_{-i} > F\left(\delta\left(1 - f\left(\delta\left(1 - f\left(\delta\right)\right)\right)\right)\right)$ if $\mathrm{mod}\left[\delta\left(1 - f\left(\delta\left(1 - f\left(\delta\right)\right)\right)\right), g\right] \neq 0$, $\mathrm{mod}\left[\delta, g\right]$ $= 0$, $\mathrm{mod}\left[\delta\left(1 - F\left(\delta\right)\right), g\right] = 0$;
$x_{-i} > f\left(\delta\left(1 - f\left(\delta\left(1 - f\left(\delta\right)\right)\right)\right)\right)$ if $\mathrm{mod}\left[\delta\left(1 - f\left(\delta\left(1 - f\left(\delta\right)\right)\right)\right), g\right] = 0$, $\mathrm{mod}\left[\delta, g\right]$ $= 0$, $\mathrm{mod}\left[\delta\left(1 - F\left(\delta\right)\right), g\right] = 0$.
Let $\left\{z^{k+1}\right\}_{k=0}^{\infty}$ be a monotonic decreasing sequence, starting with $z^1 = F\left(\delta\right)$ if $\mathrm{mod}\left[\delta, g\right] \neq 0$ or $z^1 = f\left(\delta\right)$ if $\mathrm{mod}\left[\delta, g\right] = 0$, defined by

$$z^{k+1} \equiv \begin{cases} F\left(\delta\left(1 - F\left(\delta\left(1 - z^k\right)\right)\right)\right) & \text{if } \begin{cases} \mathrm{mod}\left[\delta\left(1 - F\left(\delta\left(1 - z^k\right)\right)\right), g\right] \neq 0 \\ \mathrm{mod}\left[\delta\left(1 - z^k\right), g\right] \neq 0 \end{cases} \\ F\left(\delta\left(1 - f\left(\delta\left(1 - z^k\right)\right)\right)\right) & \text{if } \begin{cases} \mathrm{mod}\left[\delta\left(1 - F\left(\delta\left(1 - z^k\right)\right)\right), g\right] \neq 0 \\ \mathrm{mod}\left[\delta\left(1 - z^k\right), g\right] = 0 \end{cases} \\ f\left(\delta\left(1 - F\left(\delta\left(1 - z^k\right)\right)\right)\right) & \text{if } \begin{cases} \mathrm{mod}\left[\delta\left(1 - F\left(\delta\left(1 - z^k\right)\right)\right), g\right] = 0 \\ \mathrm{mod}\left[\delta\left(1 - z^k\right), g\right] \neq 0 \end{cases} \\ f\left(\delta\left(1 - f\left(\delta\left(1 - z^k\right)\right)\right)\right) & \text{if } \begin{cases} \mathrm{mod}\left[\delta\left(1 - F\left(\delta\left(1 - z^k\right)\right)\right), g\right] = 0 \\ \mathrm{mod}\left[\delta\left(1 - z^k\right), g\right] = 0 \end{cases} \end{cases}$$

which converges (in a finite number of steps) to $\overline{z}$ solution of equation (4). Then, it is straightforward that $s \in R^{\infty}$ are such that player i will never offer $x_{-i} > \overline{z}$, will accept all $x_i > \overline{z}$, and will reject all $x_i < \underline{z}$. Strategy profiles $s \in T^{\infty}$ are such that player i will never offer $x_{-i} > \overline{z}$, will accept all $x_i \geq \overline{z}$, and will reject all $x_i < \underline{z}$, where $\underline{z}$ is solution of equation (5). ■

The next proposition is a simple application of Lemma 1.

Proposition 4 *Consider a bargaining game G(δ, g). If δ is such that $(1 - g) > \delta$, then player 1's trembling-hand rationalizable expected payoffs belong to $\left[1 - \overline{z}, 1 - \underline{z}\right]$ and player 2's trembling-hand rationalizable expected payoffs belong to $\left[\underline{z}, \overline{z}\right]$.*

This proposition gives us an interval of possible player i's trembling-hand rationalizable expected payoffs, $i = 1, 2$. That is, when player i plays a trembling-hand rationalizable strategy $s_i \in T_i^{\infty}\left(\delta, g\right)$, her expected payoff belongs to this interval. The lower bound and the upper bound, as $g \rightarrow 0^+$, converge[7] to the SPE payoff or agreement of G(δ). It is

[7]Remark that, as $g \rightarrow 0^+$, the sets $R^{\infty}\left(\delta, g\right)$ and $T^{\infty}\left(\delta, g\right)$ converge, respectively, to $R^{\infty}\left(\delta\right)$ and $T^{\infty}\left(\delta\right)$.

straightforward that G(δ, g) is solvable by trembling-hand rationalizability if and only if $\underline{z} = \overline{z}$. Example 3 is an example where this condition is satisfied.

7 Conclusion

To sum up, we have shown that rationalizability can produce delayed agreements in Rubinstein's alternating-offer bargaining game. Nevertheless, there is a unique division of the cake, which can be supported by trembling-hand rationalizability. Moreover, this unique solution is the subgame perfect equilibrium of the game. Then, we have also reconsidered an extension of the alternating-offer bargaining game wherein a smallest money unit is introduced.

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